M.Math.IInd year
First semester backpaper examination
Commutative algebra — B.Sury
Answer any 6 questions.

All rings are commutative and contain 1.

Q 1.
Suppose $\theta : A \to B$ is a ring homomorphism and assume that B is faithfully flat as an A-module. Then, an A-module M is flat (respectively, faithfully flat) if and only if the B-module $M \otimes_A B$ is flat (respectively, faithfully flat).

Q 2.
Show that every finitely presented, flat A-module is projective.

Q 3.
If A is Noetherian, and M is a finitely generated A-module, prove that $Ass(M)$ is a non-empty, finite set.

Q 4.
For a field K, consider $A = K[X,Y]$. If $P_1 = (X), P_2 = (X,Y), Q = (X^2,Y)$, show that

$$P_1 \cap P_2^2 = P_1 \cap Q$$

are both primary decompositions for the ideal (X^2, XY).

Q 5.
Prove that a ring A is Artinian if it is Noetherian and all prime ideals are maximal.

Q 6.
If M is an A-module which is complete with respect to the filitration $\{M_n\}$, show that the topology induced on M by $\{M_n\}$ must be Hausdorff.

Q 7.
For local rings A, B of a field K, recall that A is said to be dominated by B if A is a subring of B and the maximal ideal of A is the contraction of the maximal ideal of B. Prove that any valuation ring C contained in a field K is maximal with respect to the partial order induced by dominance.

Q 8.
If I is an ideal in a Noetherian ring A, prove that there are finitely many prime ideals $P_1, \cdots, P_n$ such that $P_1 P_2 \cdots P_n \subset I$.
Is the above result true for the ring $C([0,1], \mathbf{R})$? Why ?

Q 9.
Prove that the Krull dimension of $K[X,Y,Z]/(XY,XZ)$ is 2.